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IMPACT ZADEH'S THEORY TO ALGEBRAIC STRUCTURES. MULTI-ADJOINT ALGEBRAS*

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Abstract. This paper presents multi-adjoint algebras as a natural and formal extension of the original algebraic structure taken into account by Zadeh in the theories of fuzzy sets and fuzzy logic.

Keywords: fuzzy sets, fuzzy logic, approximate reasoning, logic programming, formal concept analysis, rough set theory, fuzzy relation equations.

AMS Subject Classification: 03B52, 03E72, 06A06, 06B05, 08A72.

1. Introduction

From the introduction of fuzzy sets by Lotfi A. Zadeh in 1965 in his seminal paper [132], they have been developed from a theoretical and applied point of view. A quick search in Scopus shows that there exist more than 130.000 papers related to the keywords "fuzzy sets". This data shows the great impact of Zadeh's theory, not only by this seminal paper, but because his contribution and dissemination of the fuzzy philosophy around the world.

Although the original algebraic structure on the definition of fuzzy sets was the unit interval, it was quickly extended to lattices by Goguen in 1967 [67]. Although this more general structure was not exploited at the beginning, real applications have shown the importance of considering numerical truth-value sets with incomparable elements, since in many cases there exist objects without an evident relationship. For example, this structure arises when (confidence) intervals are considered as numerical truth-values instead of single values.

Moreover, the original interpretation of intersection and union of two classical sets, as minimum and maximum in the unit interval, was generalized for considering other kind of operator satisfying similar properties. The most usual operators have been triangular norms (t-norms) and triangular conorms (t-conorms) [76], whose use was originally proposed by Ulrich Höhle in 1979. Although the notion of triangular norm was introduced by Karl Menger in 1942, it was mainly considered at the beginning in the area of probabilistic metric spaces, the adoption by the fuzzy science community has been fundamental for its great dissemination.

Zadeh introduced in [133] an approximate reasoning based on a compositional rule of inferences, which generalizes modus ponens to the fuzzy case. Indeed, this reasoning was later clearly and brightly argued by Petr Hájek, giving a general definition of modus ponens in [69]

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throughout a residuated implication. This implication was associated with a left-continuous t-norm, although the commutativity and associativity are not properties required by the soundness of the fuzzy generalization of modus ponens. For this reason, the generalization to the fuzzy case of the frameworks requiring a fuzzy modus ponens, only need fuzzy implications with an adjoint conjunctor, which could be non-commutative and/or non-associative. Indeed, the approximate reasoning proposed by Zadeh was based on the max-min relational composition which is non-commutative, but has an adjoint operation.

This justification also was key for defining adjoint pairs and triples, where only the adjoint property is assumed. As a consequence, the mathematical requirements for modeling datasets are minimal and so, its adaptation is more flexible. Hence, the consideration of these kind of operators defined on general domains, such as partial ordered sets (posets), in general, or lattices, in particular, provides the algebraic structures called multi-adjoint algebras [30, 33].

Furthermore, other feature of these algebras is the possibility of considering different adjoint pairs or triples for modeling a problem, giving an extra level of flexibility. In some sense, this was also considered in the example included by Zadeh in [133], which takes into account the Łukasiewicz logic. Specifically in this example, he considered the Gödel conjunctor and the Łukasiewicz implication, although the Gödel implication and Łukasiewicz conjunctor was not used.

Therefore, multi-adjoint algebras are mathematic formulations of the natural extension of the algebraic structure originally considered by Zadeh for fuzzy sets and fuzzy logic, only considering minimal mathematical requirements for reasoning. This paper will present multi-adjoint algebras, the relationship with another general algebras, and different fuzzy frameworks in which they have usefully offered new functionalities and more flexibility.

2. Multi-adjoint algebras: definitions and properties

Adjoint triples, which are tuples composed of an adjoint conjunctor and two residuated implications, generalize t-norms and their residuated implications. These operators are more flexible since for example neither commutativity nor associativity are required, which widens considerably the application fields of adjoint triples. It is also interesting to study these operators by pairs since, for example, as it was previously commented, a residuated implication and its adjoint conjunctor are only needed for defining the modus ponens in the fuzzy setting. Moreover, only the residuated implications are required in order to define the concept-forming operators in formal concept analysis [53, 97], while the adjoint conjunctor and only one residuated implication are used to define concept-forming operators in the generalization of rough set theory given by multi-adjoint object-oriented concept lattices and multi-adjoint property-oriented concept lattices [92].

The formal definition of implications pairs, adjoint pairs and adjoint triples is given below.

Definition 2.1. [33, 39] *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings. We say that:*

- $(\&, \swarrow)$ is a phright adjoint pair with respect to P_1, P_2, P_3 if the next equivalence is satisfied, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z$$

- $(\&, \nwarrow)$ is a phleft adjoint pair with respect to P_1, P_2, P_3 if the next equivalence is verified, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$:

$$x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

- $(\swarrow, \nwarrow)$ is a phGalois implications pair with respect to P_1, P_2, P_3 if the next equivalence is satisfied, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

- $(\&, \swarrow, \nwarrow)$ is an phadjoint triple with respect to P_1, P_2, P_3 if the following double equivalence is satisfied, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

The previous equivalences, in which a conjunctor and an implication are considered, are called phadjoint properties, and represent the fuzzy extension of modus ponens.

From the previous equivalences, we obtain interesting properties related to left/right adjoint pairs and Galois implications pairs [33, 39, 40]. In the following proposition, we include the properties associated with adjoint triples.

Proposition 2.1. [[33, 39]] *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, then the following properties are satisfied:*

(1) $\&$ is order-preserving in both arguments.
(2) $\swarrow$ and $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.
(3) $\bot_1 \,\&\, y = \bot_3$, $\top_3 \swarrow y = \top_1$, for all $y \in P_2$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.
(4) $x \,\&\, \bot_2 = \bot_3$ and $\top_3 \nwarrow x = \top_2$, for all $x \in P_1$, when $(P_2, \leq_2, \bot_2, \top_2)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.
(5) $z \nwarrow \bot_1 = \top_2$ and $z \swarrow \bot_2 = \top_1$, for all $z \in P_3$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_2, \leq_2, \bot_2, \top_2)$ are bounded posets.
(6) When the supremum and the infimum exist:
   (a) $\left(\bigvee_{x' \in X} x'\right) \& \, y = \bigvee_{x' \in X} (x' \,\&\, y)$, for all $X \subseteq P_1$ and $y \in P_2$.
   (b) $\left(\bigwedge_{z' \in Z} z'\right) \swarrow y = \bigwedge_{z' \in Z} (z' \swarrow y)$, for any $Z \subseteq P_3$ and $y \in P_2$.
   (c) $x \,\&\, \left(\bigvee_{y' \in Y} y'\right) = \bigvee_{y' \in Y} (x \,\&\, y')$, for all $Y \subseteq P_2$ and $x \in P_1$.
   (d) $\left(\bigwedge_{z' \in Z} z'\right) \nwarrow x = \bigwedge_{z' \in Z} (z' \nwarrow x)$, for all $Z \subseteq P_3$ and $x \in P_1$.
   (e) $z \swarrow \left(\bigvee_{y' \in Y} y'\right) = \bigwedge_{y' \in Y} (z \swarrow y')$, for all $Y \subseteq P_2$ and $z \in P_3$.
   (f) $z \nwarrow \left(\bigvee_{x' \in X} x'\right) = \bigwedge_{x' \in X} (z \nwarrow x')$, for all $X \subseteq P_1$ and $z \in P_3$.
(7) $z \swarrow y = \max\{x \in P_1 \mid x \,\&\, y \leq_3 z\} = \min\{x \in P_1 \mid y \leq_2 z \nwarrow x\}$, for all $y \in P_2$ and $z \in P_3$.
(8) $z \nwarrow x = \max\{y \in P_2 \mid x \,\&\, y \leq_3 z\} = \min\{y \in P_2 \mid x \leq_1 z \swarrow y\}$, for all $x \in P_1$ and $z \in P_3$.
(9) $x \,\&\, y = \min\{z \in P_3 \mid x \leq_1 z \swarrow y\} = \min\{z \in P_3 \mid y \leq_2 z \nwarrow x\}$, for all $x \in P_1$ and $y \in P_2$.

Next statements are equivalent when $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$ are complete lattices. Similar results can be obtained for implications pairs and adjoint pairs.

Proposition 2.2. [[33, 39]] *Given the complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$, an arbitrary operator $\& \colon L_1 \times L_2 \to L_3$ and the mappings $\swarrow \colon L_3 \times L_2 \to L_1$, $\nwarrow \colon L_3 \times L_1 \to L_2$, defined as $z \swarrow y = \sup\{x' \in L_1 \mid x' \,\&\, y \preceq_3 z\}$ and $z \nwarrow x = \sup\{y' \in L_2 \mid x \,\&\, y' \preceq_3 z\}$, respectively, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$, the following statements are equivalent:*

(1) $(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$.
(2) For all $x \in L_1$, $X \subseteq L_1$, $y \in L_2$ and $Y \subseteq L_2$,

$$\left(\bigvee_{x' \in X} x'\right) \& \, y = \bigvee_{x' \in X} (x' \,\&\, y) \quad \text{and} \quad x \,\&\left(\bigvee_{y' \in Y} y'\right) = \bigvee_{y' \in Y} (x \,\&\, y')$$

(3) $z \swarrow y = \max\{x' \in L_1 \mid x' \,\&\, y \preceq_3 z\}$ and $z \nwarrow x = \max\{y' \in L_2 \mid x \,\&\, y' \preceq_3 z\}$ for all $x \in L_1$, $y \in L_2$ and $z \in L_3$, and $\&$ is order-preserving in both arguments.

In fuzzy logic and other fuzzy frameworks the set of operators need to be fixed from the beginning, as well as the domains in which the computations will be done. That is, an algebraic structure need to be fixed. Next, the algebraic structures associated with implications pairs, adjoint pairs and adjoint triples are presented.

Definition 2.2. [[33]] *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $i \in \{1, \dots, n\}$.*

- The tuple $(P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \&_1, \swarrow^1, \dots, \&_n, \swarrow^n)$ where $(\&_i, \swarrow^i)$ is a family of right adjoint pairs with respect to P_1, P_2, P_3 is called phright multi-adjoint algebra.
- The tuple $(P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \&_1, \nwarrow_1, \dots, \&_n, \nwarrow_n)$ where $(\&_i, \nwarrow_i)$ is a family of left adjoint pairs with respect to P_1, P_2, P_3 is called phleft multi-adjoint algebra.
- The tuple $(P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \swarrow^1, \nwarrow_1, \dots, \swarrow^n, \nwarrow_n)$ where $(\swarrow^i, \nwarrow_i)$ is a family of Galois implications pairs with respect to P_1, P_2, P_3 is called phantitone multi-adjoint algebra.
- The tuple $(P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ where $(\&_i, \swarrow^i, \nwarrow_i)$ is a family of adjoint triples with respect to P_1, P_2, P_3 is called phbiresiduated multi-adjoint algebra

The comparison with other algebras will be included in Section 4, before that, different frameworks in which these algebras have been considered will be introduced next. A detailed study of multi-adjoint algebras and their respective operators with illustrative examples can be found in [30, 33, 39].

3. Application fields

This section includes several application fields where multi-adjoint algebras and their corresponding operators play a fundamental role.

3.1. **Logic programing.** Fuzzy logic programming is a branch of fuzzy logic which has attracted the interest of many researchers in recent years. Different approaches have been proposed, such as monotonic and residuated logic programming [43, 44], fuzzy logic programming [124], possibilistic logic programming [57] and generalized annotated logic programs [75], among others, depending on the syntax of the rules and the way in which the truth-values affect the facts and the rules in each case. Among all these approaches, multi-adjoint logic programming [99] can be highlighted as a general framework embedding the different logical programming paradigms mentioned above. The main idea is to abstract the particular details of each paradigm and focus only on the minimum mathematical requirements that make possible the computation.

One important feature of this logical theory is based on the use of different implications in the rules of a same logic program, as well as general operators defined on complete lattices in the bodies of the rules. These features allow to increase the adaptability of the considered logical programs and provides the possibility of modeling a given knowledge system in a more flexible way, unlike other fuzzy logic programming frameworks. For instance, an illustrative example about a multi-adjoint logic program representing the behaviour of a motor is included in [98]. In the aforementioned paper, the multi-adjoint logic program is used to find explanations (causes) for given observations (symptoms) of the motor by means of a semantical consequence. Following the multi-adjoint philosophy, diverse implications have been assigned to the rules of the considered program. This fact let us consider different relationships among the variables depending on the "correlation" among them, for example, the relationship among the variables are different for each kind of vehicle, whose consideration could be essential to deliver a proper diagnosis of its behaviour.

Finally, it is important to mention that multi-adjoint logic programming has been developed in different lines which have increased its potential as well as studied its main properties [27, 24, 29, 42, 73, 74, 89, 94, 98, 100, 103, 104].

3.2. **Formal concept analysis.** Formal concept analysis is a theory of data analysis which identifies conceptual structures among data sets. Specifically, it is a tool to extract pieces of information from databases, which contain a set of attributes A and a set of objects B related to each other by means of a relation $R \subseteq A \times B$. These pieces of information are called concepts and a hierarchy can be established on them providing an algebraic structure called concept lattice. From the concept lattice, a mathematical development can be carried out for the conceptual analysis of data and the processing of the knowledge.

Since its introduction by B. Ganter and R. Wille [61, 127] in the eighties, formal concept analysis has become an appealing research topic and different generalizations to the fuzzy case have been proposed. Fuzzy concept lattices were firstly presented by Burusco and Fuentes-González in [17, 18], where residuated implications were not initially considered. Pollandt [114] and Bělohlávek [5] proposed independently the use of complete residuated lattices as structures for the truth degrees, being proven a representation theorem in a fuzzy framework in [6]. Later, Bělohlávek considered a fuzzy partial order on a fuzzy concept lattice instead of on an ordinary partial order in [7]. Georgescu and Popescu extended this framework to non-commutative logic and similarity in [63, 64, 65, 66]. Krajči presented the so-called generalised concept lattices in [79, 78]. Another fuzzy approach was proposed in [95, 96, 97, 101], where the philosophy of the multi-adjoint framework was applied to formal concept analysis in order to introduce multi-adjoint concept lattices.

Multi-adjoint concept lattices arise as a general approach capable of conveniently embed different fuzzy extensions of concept lattices given in the literature such as [7, 18, 64, 79] and mentioned above. With the idea of providing a general (non-commutative) environment and more flexibility into the language, the consideration of different adjoint triples or pairs with respect to a given triple of posets becomes natural in the multi-adjoint concept lattice framework [53, 97].

In this framework, different implications are used to compute the concepts of a multi-adjoint context. This fact allows to offer the possibility of considering different degrees of preference on the attributes/objects of a database. These preference degrees can be interpreted as the values of a membership function modeling a preference on the attributes and/or objects, following the semantics proposed by Zadeh, in which the values represent the intensity of preference in favor of a specific attribute/object [59]. Specifically, the problem of choosing a suitable journal to submit a scientific paper, according to the preferences of the scientist on some parameters appearing in the ISI Journal Citation Report, and by using multi-adjoint concept lattice techniques, was exemplified in [97]. Since the multi-adjoint framework support associating different adjoint triples with each object/attribute, for example, if the scientist would like to submit preferably to a journal listed under a particular category, the multi-adjoint framework allows to modify the underlying context to assign a different adjoint triple to the journals he/she is more interested in.

By last, it is convenient to emphasize other important tasks where multi-adjoint concept lattice theory has successfully been applied, such as the attribute reduction, the reduction of the size of the concept lattice, as well as the computation of reducts, that is, minimal sets of attributes containing the main information of a database [4, 31, 34, 35, 37, 38]. Another important research line in formal concept analysis is the computation of attribute implications [10, 22, 58, 84, 115, 116], which will be complemented in the multi-adjoint framework in the future [36].

3.3. **Rough sets theory.** Rough sets were introduced by Pawlak [109, 110] also in the eighties as an independent and formal tool for modeling and processing incomplete information contained in information systems. One of the most flexible extension of rough set theory to the fuzzy setting is the one presented in [41, 91], which is called multi-adjoint fuzzy rough sets. This extension implements the multi-adjoint philosophy and so, for instance, the consideration of several adjoint triples allows to assume preference among the objects as it was shown in [92, 97].

In rough set theory, datasets are represented as an information system $(X, \mathcal{A})$, or as a phdecision system $(X, \mathcal{A} \cup \{d\})$, where $X = \{x_1, \dots, x_n\}$ and $\mathcal{A} = \{a_1, \dots, a_m\}$ are finite non-empty sets of objects and attributes, respectively, and $d \notin \mathcal{A}$ is called the decision attribute. Each a in $\mathcal{A}$ corresponds to a mapping $\bar{a}\colon X \to V_a$, where V_a is the value set of a over X. On these mathematical representations of (relational) datasets different operators and methodologies are developed. The first most important operator in rough set theory is the so-called B-indiscernibility relation, which relates each pair of objects through the given attributes in $B \subseteq \mathcal{A}$. In the fuzzy multi-adjoint framework this relation does not need to satisfy the transitivity property, unlike the classical case, and it is based on tolerance relations and an aggregator operator. The other most important properties are the lower and upper approximations, which are extended with the use of adjoint pairs. As in formal concept analysis the attribute selection is an important task, although in rough set theory the philosophy is completely different. Here, we are looking for minimal subsets of attributes preserving the discernibility among the objects. Different papers have been published studying the relationship between both notions of reducts [11, 12, 14]. In [41], general and useful measures were introduced for detecting rough set reducts. Attribute selection has also been extended to reduce both, the set of attributes and the sets of objects, presenting the notion of bireduct [120, 121]. Bireducts were introduced in rough set theory as one of successful solutions for reducing the number of attributes by preventing the occurrence of incompatibilities and eliminating existing noise in the original data. These have been generalized with the use of tolerance and equivalence relations [15] and also applied to formal concept analysis [12, 13]. Thus, it would be interesting the study of bireducts in the fuzzy multi-adjoint framework. Other research lines in which the multi-adjoint philosophy can offer new advances is in the computation of decision rules [111] and in new object classification [88, 123], which will also be studied in the near future.

3.4. **Isotone concept lattices.** As we highlighted above, rough set theory and formal concept analysis are fundamental tools for extracting information from relational knowledge systems. In the classical case, these two important tools have been related [80, 86, 130, 131] and consequently, the obtained results for rough set theory can be applied to formal concept analysis and vice versa [20, 125]. In the fuzzy case, the derivation operators have been related in a fuzzy level in [55, 65], although both extensions require of a negation operator. This relationship has been mainly based on the consideration of both sets -the set of objects and the set of attributes- of an information system in rough set theory, instead of directly applying the upper and lower approximations to the indiscernibility relation. From this idea, the property-oriented and object-oriented concept lattices arise in the classical case [62, 130]. The concept forming operators in these frameworks are called necessary and possibility operators, which also shows their relationship with the usual modal operators and possibility theory [56]. Moreover, these operators form isotone Galois connections (also called adjunctions) instead of antitone Galois connections [48], as the concept forming operators in formal concept analysis form. Other interesting papers on isotone concept lattices and isotone Galois connections are [9, 19, 77].

Multi-adjoint property and object-oriented concept lattices were introduced in [92] as an generalization of the current frameworks, with the main goals of reducing unnecessary mathematical requirements, providing a more flexibility setting and giving extra capabilities for obtaining information for datasets. These frameworks have been fundamental for example in the study of the solvability of fuzzy relation equations [50, 52] and in fuzzy mathematical morphology [1, 2, 90].

As in the multi-adjoint concept lattice framework presented in [97], once again, the possibility of considering several conjunctors and implications in order to compute the concepts of a same lattice allows to establish different degrees of preference on the attributes/objects of a database. This fact is exemplified in [92], where a set of countries (objects) and a set of polluting gases (attributes) are considered in order to know what country contaminates the least. If there are countries with less possibilities of decreasing their contamination level, because of their third world condition, the underlying context can be modified to assign preference to these countries, reasoning that we would prefer one of these countries win our contest, although this preferability is not an imposition. In addition, the theory developed in [92] shows the flexibility of multi-adjoint property-oriented and object-oriented concept lattices to be applied in problems where a threshold cannot be exceeded, such as the pollution problem of the aforementioned example, economic problems about budget, construction of buildings, medical diagnosis where certain levels of cholesterol, sugar, bilirubin, etc. should not be overcome.

3.5. **Fuzzy relation equations.** Fuzzy relation equations are associated with the composition of fuzzy relations and they were introduced by Elie Sanchez in the seventies, in order to simulate the relationship between cause and effect in medical diagnosis problems [118, 119]. Since then, fuzzy relations have beed used to investigate theoretical and applied aspects of fuzzy set theory such as decision making and arithmetic of fuzzy numbers [112], knowledge engineering [49], treatment of images and videos [70, 71, 87, 106, 107, 108], fuzzy control [21], optimization problems [81, 82, 102, 129], wireless communication management models [129], among others.

Following the philosophy of the multi-adjoint framework, multi-adjoint relation equations [50, 51, 52] arise as a generalization of the usual fuzzy relation equations [45, 49, 113]. In these last papers, the relationship with concept lattices was fundamental for obtaining useful properties, such as the characterization of the whole set of solutions of a multi-adjoint relation equation.

As in the previous application fields, the multi-adjoint nature of these equations allows to simulate knowledge systems with uncertainty in a more flexible way. For example, multi-adjoint relation equations have been used as a support mechanism for the negotiations of sellers in [23]. Specifically, a fuzzy rules system for the reasoning process of a seller, which indicates if the seller should accept/reject the buyer's offer, is considered. This fuzzy rules system is interpreted as a multi-adjoint logic program for the decision making in the negotiation process. The weights of the rules of the aforementioned logic program are obtained solving a multi-adjoint fuzzy relation equation. This fact shows that fuzzy relation equations and fuzzy logic programming are theories that complement and enrich each other. This relationship has also been highlighted in [3, 117], in which a novel approach based on fuzzy logic and fuzzy relation equations has been presented for cause-effect variable analysis for companies, with the main goal of knowing the main actions the considered company must perform for increasing their benefits.

More advances in multi-adjoint relation equations, such as the acquisition of an approximate answer for non-solvable equations, the consideration of other general structures, and the resolution of equations where the variables have a bipolar character, can be found in the following papers [25, 26, 28, 54, 93].

4. A comparative study with general algebraic structures

There exist many other structures in the literature in the great and big world of fuzzy sets and fuzzy logic. Hence, it is needed to compare multi-adjoint algebras with them or, at least, with some of the most used and general. This section includes four diagrams offering an overview on the relations between multi-adjoint algebras and other algebraic structures such as adjointness algebras [105], sup-preserving aggregations [8], quantales [16, 72], u-norms [83], semi-uninorms and uninorms [60, 85, 122, 126, 128], and general implications, such as implication structures [134] and the ones considered in extended-order algebras [46, 47, 68]. For a suitable understanding of these diagrams, it should be noted that algebraic structures joined by arrows with a single tip indicate that the structure displayed at the initial node is a particular case of the algebraic structure located in the final node. Those algebraic structures connected by arrows with double tips are equivalent. The flow of the arrows in the diagrams is, more or less, bottom-up. In addition, these diagrams are transitive and so the arrows can be deduced by the transitivity of the represented relations are not drawn.

Fig.1 displays a diagram showing the relationships without restrictions. Fig.2, 3 and 4 represent diagrams considering the conditions required in the frameworks where multi-adjoint concept lattices, fuzzy multi-adjoint rough sets and multi-adjoint logic programming are defined, respectively. The differences among the diagrams are highlighted by arrows with different colors. A detailed study on the comparative of multi-adjoint algebras with the aforementioned non-commutative algebraic structures can be found in [32, 33].

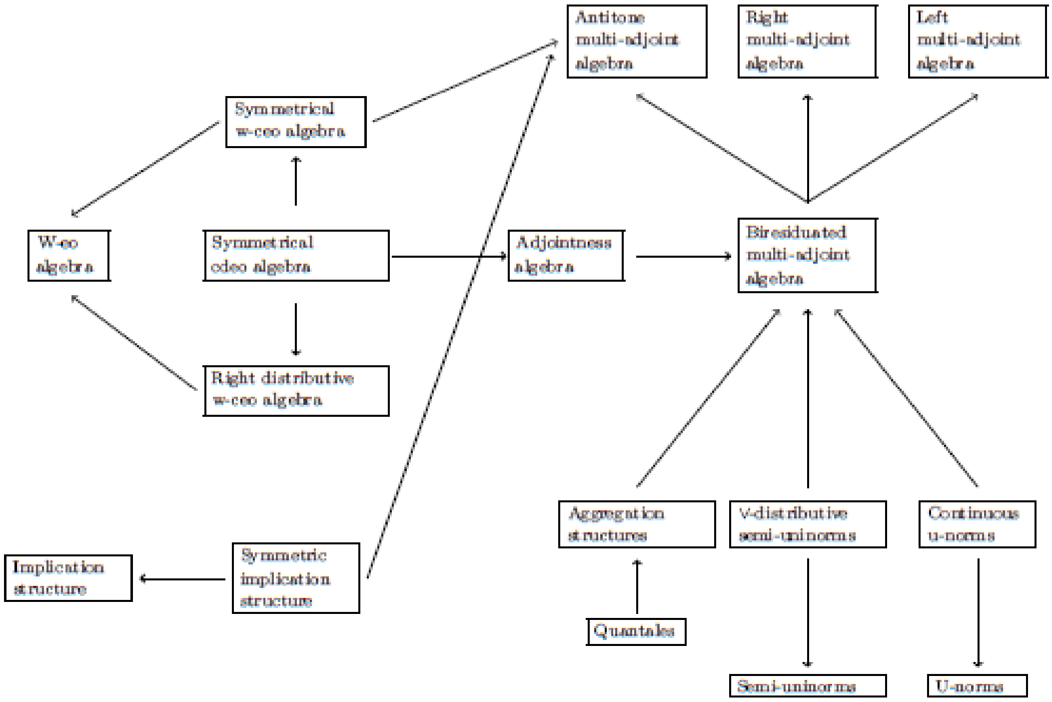

Figure 1. General comparative diagram.

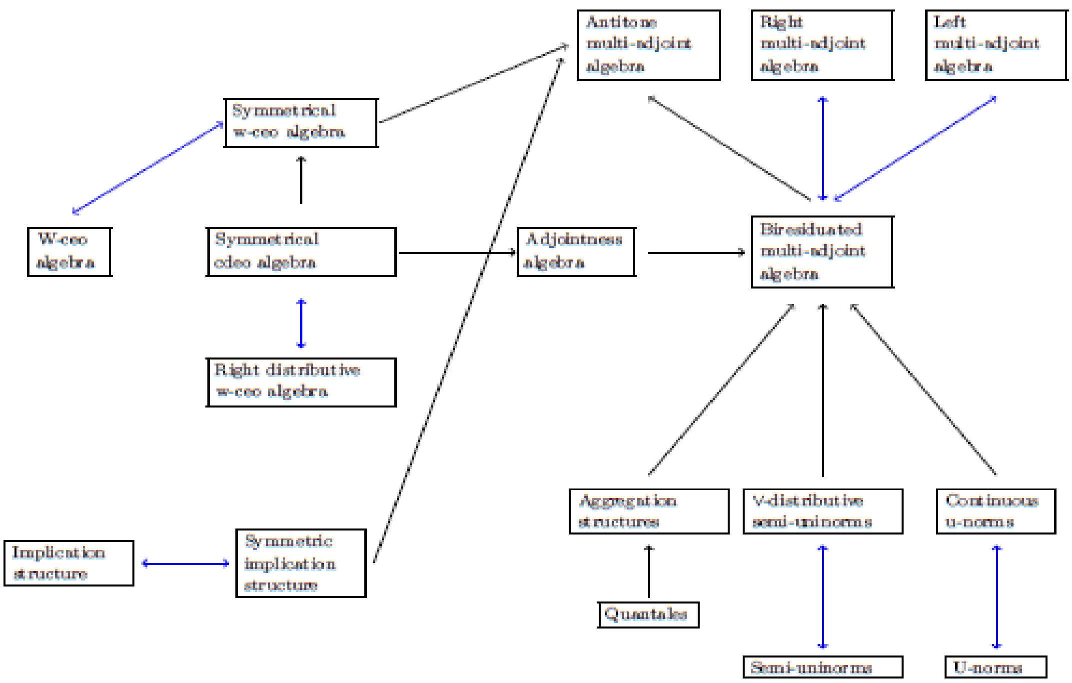

Figure 2. Comparative diagram in the multi-adjoint concept lattices framework.

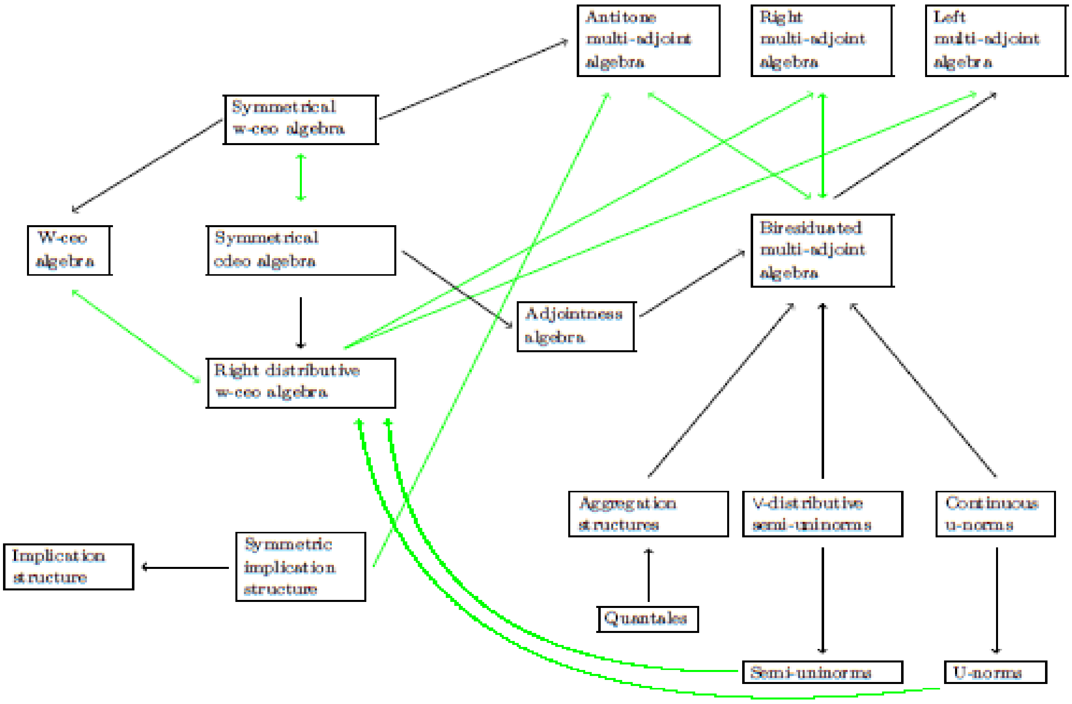

Figure 3. Comparative diagram in the multi-adjoint rough sets.

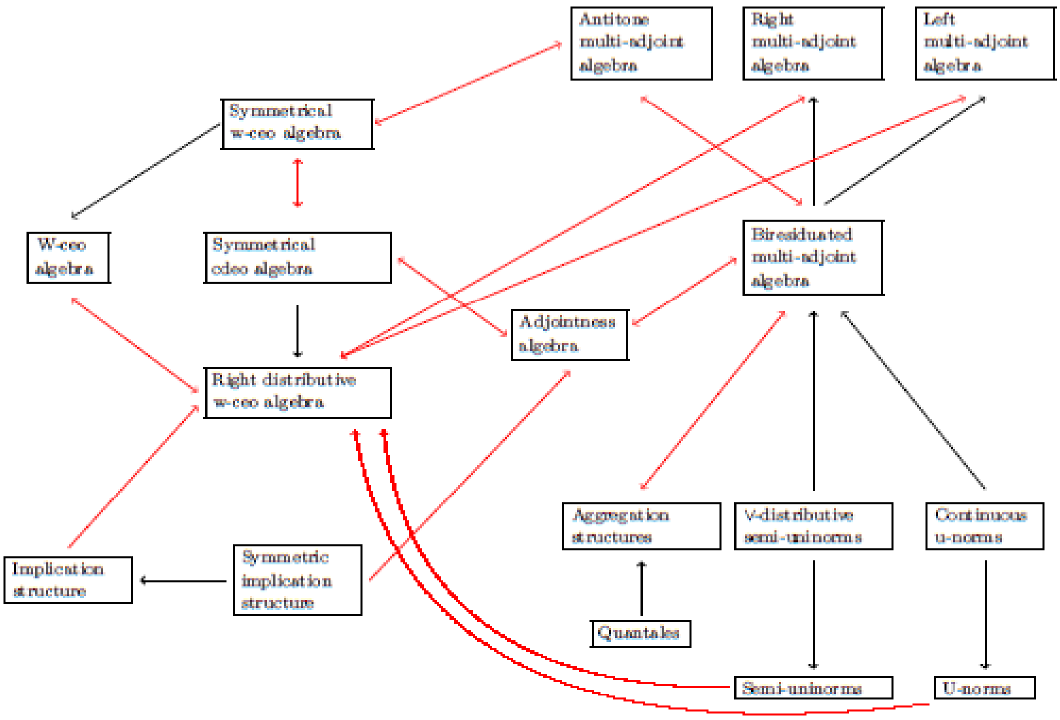

Figure 4. Comparative diagram in the multi-adjoint logic programming framework.

5. Conclusions and future work

This paper has highlighted the importance of multi-adjoint algebras as an formal and natural extension of the original algebraic structure taken into account by Zadeh in his theories of fuzzy sets and fuzzy logic. The most representative properties have been included, as well as the comparison with other general algebraic structures in different fuzzy frameworks. These contributions have been complemented with the inclusion of five different fuzzy frameworks in which multi-adjoint algebras have played a fundamental role for offering them an extra level of flexibility and adaptability to real problems (datasets).

In the future, more properties and fuzzy frameworks will be studied to increase the power of multi-adjoint algebras and so, the Zadeh's legacy.

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